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COMMENT

On the classical mechanics of a non-relativistic superparticle

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Abstract. A classical Lagrangian function for a system with even and odd degrees of freedom is proposed and studied. The quantisation is performed without the appearance of non-trivial constraints and it leads to the previously defined supersymmetric Schrödinger equation.

The most significant property of the three-dimensional non-relativistic supersymmetry is the appearance of Lorentz (e.g., relativistic) symmetry as a dynamical symmetry. In a recent paper (Sokatchev and Stoyanov 1986) a supersymmetrically invariant equation has been proposed in a form analogous to the quantum mechanical Schrödinger equation. It has been defined by the requirement to remain invariant under the action of a three-dimensional superalgebra which contains the Euclidean group rather than the Lorentz group. In this equation which we call a supersymmetric Schrödinger equation the time variable plays the role of a parameter. It turns out that the equations of motion for the bosonic and the fermionic components of the superwavefunction are the relativistically invariant Klein-Gordon and Dirac equations respectively. Thus Lorentz invariance appears as a dynamical symmetry of the considered supersymmetric system as a quantum mechanical one.

This property of the non-relativistic supersymmetry also survives in cases with interaction. As is shown (Aneva *et al* 1987) one can redefine the supersymmetric Schrödinger equation in such a way that it contains different interaction terms and still remains invariant under supersymmetry.

As is seen from the paper by Sokatchev and Stoyanov (1986) the supersymmetric Schrödinger equation has in fact been *postulated.* But on the other hand, we can try to obtain the same equation by quantising a suitable classical mechanics system which together with the usual even dynamical variables (the canonically conjugated coordinates and momenta) also includes odd variables. The quantisation of the classical system in our opinion should be done in such a way that no non-trivial constraints appear.

In the present paper we propose a solution to this problem defining a proper classical Lagrangian function for an object with even and odd degrees of freedom which we call a non-relativistic superparticle. From this Lagrangian we find with the help of the usual commutation relations between the coordinates and the momenta that it leads to the Hamiltonian operator in the supersymmetric Schrödinger equation. The main result of this investigation is that the general solution of the classical

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superparticle equations of motion contains harmonic terms with Grassmann (or nilpotent) amplitudes. This means actually that the motion of the superparticle should not be interpreted as free. Nevertheless, we find after quantisation that the supersymmetric Schrödinger equation describes free superfields.

For completeness we begin with the discussion of the above-mentioned non-relativistic superalgebra. It is the supersymmetric extension of the Lie algebra of the three-dimensional Euclidean group $T_3 \times O(3)$ or SU(2). It contains the following even generators: the three O(3) generators I_k, and the three translations P_k, $(k=1,2,3)$; the odd generators Q_α, $\alpha=1,2$ form an SU(2) complex spinor. These generators satisfy the following commutation and anticommutation relations:

$$
\begin{aligned}
&[I_k, I_l] = \mathrm{i}\varepsilon_{kln} I_n \\
&[I_k, P_l] = \mathrm{i}\varepsilon_{kln} P_n \\
&[P_k, P_l] = 0 \qquad [I_k, Q_\alpha] = -\tfrac{1}{2}(\sigma_k Q)_\alpha \\
&\{Q_\alpha, Q_\beta\} = \mathcal{N}(\sigma^l \varepsilon)_{\alpha\beta} P_l \qquad [P_k, Q_\alpha] = 0.
\end{aligned}
\tag{1}
$$

Here σ^k denote the Pauli matrices, ε_{kln} is the fully antisymmetric tensor ($\varepsilon_{123}=1$); $\varepsilon = \mathrm{i}\sigma^2$ is the two-dimensional metric tensor $\varepsilon^2=-1$. With the help of $\varepsilon_{\alpha\beta}$ we raise and lower the spinor indices. The constant N remains unspecified.

As usual the representation of this superalgebra is constructed in the superspace $(x_k, \theta Q)$ where x_k are the coordinates of the three-dimensional Euclidean space R^3 and θ_α are two-component (complex) spinors, Grassmann variables

$$
\{\theta_\alpha, \theta_\beta\} = 0 \qquad \alpha, \beta = 1, 2. \tag{2}
$$

The generators P_k and Q_α are realised as differential operators in the superspace:

$$
\begin{aligned}
&P_k = -\mathrm{i}\frac{\partial}{\partial x_k} \qquad k=1,2,3 \\
&Q_\alpha = \mathrm{i}\frac{\partial}{\partial \theta^\alpha} + \mathrm{i}\frac{\mathcal{N}}{2}(\sigma^k\theta)_\alpha P_k.
\end{aligned}
\tag{3}
$$

A spinor covariant derivative $\mathscr{D}_\alpha$ can be defined

$$
\mathscr{D}_\alpha = \mathrm{i}\frac{\partial}{\partial \theta^\alpha} - \mathrm{i}\frac{\mathcal{N}}{2}(\sigma^k\theta)_\alpha P_k \tag{4}
$$

which satisfies the following anticommutation relations:

$$
\{\mathscr{D}_\alpha, \mathscr{D}_\beta\} = -\mathcal{N}(\sigma^k\varepsilon)_{\alpha\beta} P_k \qquad \{Q_\alpha, \mathscr{D}_\beta\} = 0. \tag{5}
$$

The operator

$$
K = \frac{4}{\mathcal{N}^2}\mathscr{D}^\alpha \mathscr{D}_\alpha \tag{6}
$$

plays the role of a kinetic energy operator, i.e. a supersymmetric Hamiltonian operator in the free supersymmetric Schrödinger equation (Sokatchev and Stoyanov 1986, Aneva *et al* 1987). In the quantum case it has the form:

$$
K(P, \pi, \theta) = \frac{4}{\mathcal{N}^2}\pi^\alpha \pi_\alpha + \frac{4\mathrm{i}}{\mathcal{N}}\pi^\alpha(\sigma^k\theta)_\alpha P_k - P^2\theta^\alpha\theta_\alpha \tag{7}
$$

where $P_k = -\mathrm{i}\partial/\partial x_k$, $\pi^\alpha = -\mathrm{i}\partial/\partial\theta_\alpha$ are differential operators. In the superspace we consider, the quantities (P^k, x_l) and $(\pi^\alpha, \theta_\beta)$ are two pairs of canonically conjugated elements which satisfy commutation and anticommutation relations respectively

$$\begin{aligned} [P^k, x_l] &= -\mathrm{i}\delta^k_l \\ \{\pi^\alpha, \theta_\beta\} &= -\mathrm{i}\delta^\alpha_\beta. \end{aligned} \tag{8}$$

Now we want to discuss the classical mechanics of a superparticle. Having in mind the operator $K(P, \pi, \theta)$ we are ready to reconstruct an expression for the classical Hamiltonian, which after quantisation will lead to our supersymmetric Hamiltonian (7). We postulate

$$H_{\mathrm{cl}} = \frac{4}{\mathcal{N}^2}\pi^\alpha\pi_\alpha + \frac{4\mathrm{i}}{\mathcal{N}}\pi^\alpha(\sigma^k\theta)_\alpha p_k - p^2\theta^\alpha\theta_\alpha. \tag{9}$$

The coordinates (x_k, p^l) and $(\theta_\alpha, \pi^\beta)$ are the canonically conjugated dynamical variables in the Hamiltonian formulation of a classical mechanics with bosonic and fermionic degrees of freedom. They satisfy the corresponding Poisson brackets

$$\begin{aligned} \{p^k, x_l\}_{\mathrm{P}} &= -\delta^k_l \\ \{\pi^\alpha, \theta_\beta\}_{\mathrm{P}} &= -\delta^\alpha_\beta \end{aligned} \tag{10}$$

which after quantisation will go into the commutation and anticommutation relations (8) according to the rule

$$\mathrm{i}\{\ \}_{\mathrm{P}} \to \begin{matrix}\text{commutator}\\ \text{anticommutator.}\end{matrix}$$

We use the following definition of Poisson brackets in the presence of anticommuting variables (Casalbuoni 1976, Fradkin *et al* 1978):

$$\begin{aligned} \{E_1, E_2\} &= \left(\frac{\partial E_1}{\partial q_i}\frac{\partial E_2}{\partial p^i} - \frac{\partial E_2}{\partial q_i}\frac{\partial E_1}{\partial p^i}\right) + \left(\frac{\partial E_1}{\partial \theta_\alpha}\frac{\partial E_2}{\partial \pi^\alpha} - \frac{\partial E_2}{\partial \theta_\alpha}\frac{\partial E_1}{\partial \pi^\alpha}\right) \\ \{O, E\}_{\mathrm{P}} = -\{E, O\}_{\mathrm{P}} &= \left(\frac{\partial O}{\partial q_i}\frac{\partial E}{\partial p^i} - \frac{\partial E}{\partial q_i}\frac{\partial O}{\partial p^i}\right) - \left(\frac{\partial O}{\partial \theta_\alpha}\frac{\partial E}{\partial \pi^\alpha} + \frac{\partial E}{\partial \theta_\alpha}\frac{\partial O}{\partial \pi^\alpha}\right) \\ \{O_1, O_2\}_{\mathrm{P}} &= \left(\frac{\partial O_1}{\partial q_i}\frac{\partial O_2}{\partial p^i} + \frac{\partial O_2}{\partial q_i}\frac{\partial O_1}{\partial p^i}\right) - \left(\frac{\partial O_2}{\partial \theta_\alpha}\frac{\partial O_1}{\partial \pi^\alpha}\right) \end{aligned} \tag{11}$$

where E and O denote even and odd variables respectively. To describe the classical mechanics of a superparticle we have to discuss the Hamiltonian equations of motion which describe the time evolution of the dynamical variables q_A and p_A. In the presence of Grassmann variables these equations have the form (Mañes and Zumino 1985)

$$\begin{aligned} \dot{q}_A &= \sigma(A)\frac{\partial H_{\mathrm{cl}}}{\partial p_A} \qquad A = (k, \alpha) \\ \dot{p}_A &= -\frac{\partial H_{\mathrm{cl}}}{\partial q_A} \qquad \sigma(k) = 1 \qquad \sigma(\alpha) = -1. \end{aligned} \tag{12}$$

In our case we have, respectively,

$$\begin{aligned} \dot{x}_k &= \frac{\partial H_{\mathrm{cl}}}{\partial p^k} \qquad \dot{\theta}_\alpha = -\frac{\partial H_{\mathrm{cl}}}{\partial \pi^\alpha} \\ \dot{p}_k &= -\frac{\partial H_{\mathrm{cl}}}{\partial x_k} \qquad \dot{\pi}_\alpha = -\frac{\partial H_{\mathrm{cl}}}{\partial \theta^\alpha}. \end{aligned} \tag{13}$$

After substituting the expression for H_{cl} these equations become:

$$\begin{aligned} &\dot{p}_k = 0 \qquad \dot{x}_k = (4\mathrm{i}/\mathcal{N})\pi^\alpha(\sigma_k\theta)_\alpha - 2p_k\theta^\alpha\theta_\alpha \\ &\dot{\theta}_\alpha = -\frac{8}{\mathcal{N}^2}\pi_\alpha - \frac{4\mathrm{i}}{\mathcal{N}}(\sigma^l\theta)_\alpha p_l \\ &\dot{\pi}_\alpha = -\frac{4\mathrm{i}}{\mathcal{N}}(\sigma^l\pi)_\alpha p_l + 2p^2\theta_\alpha. \end{aligned} \tag{14}$$

The system of differential equations (14) can be easily solved and its general solution is found to be

$$\begin{aligned} x_k(t) &= x_k^0 + \rho^\alpha\rho_\alpha p_k t - \tfrac{1}{2}\mathrm{i}\mathcal{N}\zeta\sigma_k\rho \cos\left(\frac{8}{\mathcal{N}}\sqrt{p^2}t\right) - \frac{\mathcal{N}}{2}\zeta\sigma_k\sigma^n p_n\rho \sin\left(\frac{8}{\mathcal{N}}\sqrt{p^2}t\right) \\ \theta_\alpha(t) &= \zeta_\alpha + \rho_\alpha \cos\left(\frac{8}{\mathcal{N}}\sqrt{p^2}t\right) + \mathrm{i}\frac{p_k}{\sqrt{p^2}}(\sigma^k\rho)_\alpha \sin\left(\frac{8}{\mathcal{N}}\sqrt{p^2}t\right) \end{aligned} \tag{15}$$

with x_k^0, p_k, ρ^α, ζ_α arbitrary (ρ, ζ Grassmann) constants fixed by the initial conditions. It is evident from the expressions (15) that $x_k(t)$ is a superposition of the general solution of the equation of motion of a free particle and nilpotent harmonic terms. We observe the similarity of the form of this solution with a given mode of the string. We note also that the solution for $\theta_\alpha(t)$ does not contain a term linear in time. As is readily seen from the Hamiltonian equations the quantity p_k is a constant of motion. Writing the classical Hamiltonian function in the form

$$2H_{cl} = \dot{\theta}^\alpha\pi_\alpha + \dot{x}^k p_k \tag{16}$$

we can easily convince ourselves that

$$\mathrm{d}H_{cl}/\mathrm{d}t = 0. \tag{17}$$

The same is valid for the spinor quantity

$$Q_\alpha = -\pi_\alpha + \mathrm{i}\frac{\mathcal{N}}{2}(\sigma^k\theta)_\alpha p_k. \tag{18}$$

Thus in the classical case three quantities are constants of motion: P_k, H and Q_α. In the quantum case the first two remain automatically constants of motion, but one has to choose only one of the Q_α since they do not anticommute.

Now we are going to construct a Lagrangian $L(q_A, \dot{q}_A)$, $q_A = (x_k, \theta_\alpha)$ for a superparticle as a function of the coordinates q_A and the velocities $\dot{q}_A$ from the requirement for superinvariance. Under the transformation generated by Q_α (with a parameter $\varkappa^\alpha$) the coordinates x_k and θ_α transform as follows:

$$\begin{aligned} x_k^s &= x_k + \mathrm{i}(\mathcal{N}/2)\varkappa\sigma_k\theta \\ \theta_\alpha^s &= \theta_\alpha - \varkappa_\alpha. \end{aligned} \tag{19}$$

As is readily seen the quantity $\dot{\theta}_\alpha$ is itself an invariant under the supersymmmetry transformation (19). Out of x_k, θ_α, $\dot{x}_k$ and $\dot{\theta}_\alpha$ we can construct the quantity

$$Y_k = \dot{x}_k + \mathrm{i}\frac{\mathcal{N}}{2}\theta\sigma_k\dot{\theta}. \tag{20}$$

It is easy to show that this quantity also remains invariant under the supersymmetry transformation (19).

We are now ready to write the most general Lagrangian function invariant under the transformations generated by our superalgebra. It will be a polynomial of Y_k and will contain a term $\dot\theta^\alpha\dot\theta_\alpha$

$$L(q, \dot q) = a_k Y_k + bY_k^2 + \lambda\dot\theta^\alpha\dot\theta_\alpha. \tag{21}$$

A simple dimensional analysis shows that the constant b in (21) should have the dimension of mass ($[l]^{-1}$). But our initial quantum theory does not contain dimensional parameters and since this discussion is justified, namely by the correspondence of the Lagranian (21) to the already found quantum Hamiltonian, we conclude that a term $\sim bY_k^2$ should not be included in the Lagrangian function.

We thus propose

$$L(q, \dot q) = -a_k Y_k - \tfrac{1}{16}\mathcal{N}^2\dot\theta^\alpha\dot\theta_\alpha \tag{22}$$

to be the Lagrangian of a free superparticle and we are going to show that this Lagrangian (with the choice of $\lambda = -\frac{1}{16}\mathcal{N}^2$) leads to our classical Hamiltonian (9). This Lagrangian will describe the dynamics of a massless superparticle since it does not contain a mass term.

We first define the conjugated momenta

$$\begin{aligned} p_k &\equiv \frac{\partial L}{\partial \dot x_k} = -a_k \qquad \Pi_k \equiv \frac{\partial L}{\partial \dot a_k} = 0 \\ \pi_\alpha &\equiv \frac{\partial L}{\partial \dot\theta^\alpha} = -\tfrac{1}{8}\mathcal{N}^2\dot\theta_\alpha - \tfrac{1}{2}\mathrm{i}\mathcal{N}(\sigma^k\theta)_\alpha p_k. \end{aligned} \tag{23}$$

Then we can pass to the Hamiltonian formulation of the superparticle dynamics according to the well known general rules

$$H = \dot x^k p_k + \dot\theta^\alpha\pi_\alpha - L. \tag{24}$$

Evidently the Lagrangian $L(q, \dot q)$ given by (22) is such that from the definition of the momenta a set of algebraic constraints follows

$$\chi_k \equiv (p_k + a_k) = 0 \qquad \Pi_k = 0. \tag{25}$$

It is easy to show that Π_k cannot be expressed by any other of the dynamical variables. Thus the constraints, although not being primary, are identically vanishing

$$p_k = -a_k \qquad \Pi_k = 0. \tag{26}$$

These relations are actually trivial—in fact a_k plays the role of a Lagrangian multiplier.

Expressing $\dot\theta_\alpha$ from the third equation in (23) and substituting into

$$H = \dot x^k p_k + \dot\theta^\alpha\pi_\alpha - L \equiv -\tfrac{1}{16}\mathcal{N}^2\dot\theta^\alpha\dot\theta_\alpha$$

we obtain

$$H = (4/\mathcal{N}^2)\pi^\alpha\pi_\alpha + (4\mathrm{i}/\mathcal{N})\pi^\alpha(\sigma^l\theta)_\alpha p_l - p^2\theta^\alpha\theta_\alpha$$

which coincides with the expression for our classical Hamiltonian.

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